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Richard and Jan-Ake have both made general statements objecting to parts of my papers, stating my states are not correct and now well defined. They have not pointed to equations, so I will guess.

I will write out the details and hope this settles the questions.

Density operators: Justification of Eq.(2) and (3),

Entangled Eq.(2)

My density operators of [Spin with hyper-helicty](), Eq.(2) and (3), use nothing new and any objections must also question the veracity of the usual states that we all use, for example, GHSZ use then, Appendix B, Eqs.(B2a and B2b). These are the states I use. Explicitly GHSZ Eqs.(A2a,A2b) are,

$$\left|+\right\rangle_i = \begin{pmatrix} \cos\theta e^{-i\phi/2} \\ \sin\theta e^{i\phi/2} \end{pmatrix}; \qquad \left|-\right\rangle_i = \begin{pmatrix} -\sin\theta e^{-i\phi/2} \\ \cos\theta e^{i\phi/2} \end{pmatrix}$$

The GHSZ Eq(B1) is the singlet, which is the same as my [Spin with hyper-helicty]() Eq(1),

$$\left|\Psi_{12}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|+\right\rangle_1\left|-\right\rangle_2 - \left|-\right\rangle_1\left|+\right\rangle_2\right)$$

This is a pure entangled states which give the state operator by the outer product of the singlet,

$$\rho_{12} = \left|\Psi_{12}\right\rangle\left\langle\Psi_{12}\right| = \frac{1}{2}\begin{pmatrix} \left|+\right\rangle_1\left|-\right\rangle_2\left\langle+\right|_1\left\langle-\right|_2 + \left|-\right\rangle_1\left\langle+\right|_2\left\langle-\right|_1\left|+\right\rangle_2 \\ -\left|+\right\rangle_1\left|-\right\rangle_2\left\langle-\right|_1\left\langle+\right|_2 - \left|-\right\rangle_1\left\langle+\right|_1\left|+\right\rangle_2\left\langle-\right|_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \left(\left|+\right\rangle\left\langle+\right|\right)_1\left(\left|-\right\rangle\left\langle-\right|\right)_2 + \left(\left|-\right\rangle\left\langle-\right|\right)_1\left(\left|+\right\rangle\left\langle+\right|\right)_2 \\ -\left(\left|+\right\rangle\left\langle-\right|\right)_1\left(\left|-\right\rangle\left\langle+\right|\right)_2 - \left(\left|-\right\rangle\left\langle+\right|\right)_1\left(\left|+\right\rangle\left\langle-\right|\right)_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & +1 & -1 \\ 0 & -1 & +1 \\ 0 & 0 & 0 \end{pmatrix}$$

This should answer Richard's question about what $\left|\pm\right\rangle\left\langle\pm\right|, \quad \left|\pm\right\rangle\left\langle\mp\right|$ means, e.g. these are pure states,

Polarizations

$$\left|+\right\rangle\left\langle+\right| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \left|-\right\rangle\left\langle-\right| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};$$

Coherences

$$\left|+\right\rangle\left\langle-\right| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \left|-\right\rangle\left\langle+\right| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};$$

Note, the state operator can be expressed in terms of Pauli Spin operators,

$$\rho_{12} = \frac{1}{4}\left(I^1 \otimes I^2 - \sigma^1 \cdot \sigma^2\right) = \frac{1}{4}\left(I^1 \otimes I^2 - \sigma_x^1 \otimes \sigma_x^2 - \sigma_y^1 \otimes \sigma_y^2 - \sigma_z^1 \otimes \sigma_z^2\right)$$

$$= \frac{1}{4}\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = |\Psi_{12}\rangle\langle\Psi_{12}|$$

My equation Eq.(2) is verified as

$$\rho_{12} = \frac{1}{4}\left(I^1 \otimes I^2 - \sigma^1 \cdot \sigma^2\right)$$

This is equivalent the usual singlet in GHSZ Eq.(4), $|\Psi_{12}\rangle$ except I use the state operator.

I hope all is clear: pure state, nothing new, regular qm like we all use, but I use $|\Psi_{12}\rangle\langle\Psi_{12}|$

Product state Eq.(3)

$$\rho_{12} \to \frac{1}{4}\left(I^1 - \sigma_z^1\right) \otimes \left(I^2 + \sigma_z^1\right) = \frac{1}{4}\left(I^1 I^2 + I^1 \sigma_z^2 - \sigma_z^1 I^2 - \sigma_z^1 \sigma_z^2\right) =$$

$$\frac{1}{4}\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \left(|+\rangle\langle+|\right)_1 \left(|-\rangle\langle-|\right)_2$$

This is a pure product state conserving angular momentum. Upon separation there is no entanglement, just a product state,

$$\rho_{12} \to \frac{1}{4}\left(I^1 \mp \sigma_z^1\right) \otimes \left(I^2 \pm \sigma_z^1\right) = \left(|\pm\rangle\langle\pm|\right)_1 \left(|\mp\rangle\langle\mp|\right)_2$$

Expectation values

Spin with hyper-helicty Eq. (4) gives the correlation between two operators,

$$E(A,B) = \langle AB \rangle = \mathrm{Tr}\left(A^\dagger B \rho\right)$$

This is the quantum analogue of the classical correlation, see Bell, Eq.(2). If you do not like that, start with a normed Banach space over an algebra of operators that are of trace class and form a convex set.

Then define the inner product as above to form a Hilbert-Schmidt space of operators. See von Neumann's book of 1936. The operators usually have some symmetry, like SU(2), and are trace class, like Pauli matrices. Pure states are extremal points on the convex set, and mixed states are interior points. All this is standard stuff I just mention for completeness.

This leads easily to Eq.(7), the usual correlation of $-\mathbf{a}\cdot\mathbf{b}=-\cos(\theta_a-\theta_b)$, which I will work out in detail,

$$E(a,b)=\underset{12}{\mathbf{Tr}}\left(\sigma_a^1\sigma_b^2\rho_{12}\right)=\mathbf{a}\cdot\left\langle\sigma^1\sigma^2\right\rangle\cdot\mathbf{b}$$

where $\sigma_a^i=\sigma^i\cdot\mathbf{a}$

$$\left\langle\sigma^1\sigma^2\right\rangle=\underset{12}{\mathbf{Tr}}\left(\sigma^1\sigma^2\rho_{12}\right)=\frac{1}{4}\underset{12}{\mathbf{Tr}}\left(\sigma^1\sigma^2\left(I_1\otimes I_2-\sigma_1\cdot\sigma_2\right)\right)$$

$$=\frac{1}{4}\underset{12}{\mathbf{Tr}}\left(\sigma^1\sigma^2\left(I^1\otimes I^2-\sigma^1\cdot\sigma^2\right)\right)=\frac{1}{4}\underset{12}{\mathbf{Tr}}\left(\left(\sigma^1\otimes\sigma^2-\sigma^1\sigma^1\cdot\sigma^2\sigma^2\right)\right)$$

$$=\frac{1}{4}\left(\underset{1}{\mathbf{Tr}}\sigma^1\otimes\underset{2}{\mathbf{Tr}}\sigma^2-\underset{1}{\mathbf{Tr}}\left(\sigma^1\sigma^1\right)\cdot\underset{2}{\mathbf{Tr}}\left(\sigma^2\sigma^2\right)\right)$$

note Pauli spin matrices are traceless, so $\underset{i}{\mathbf{Tr}}\sigma^i=0$ and so

$$\left\langle\sigma^1\sigma^2\right\rangle=-\frac{1}{4}\underset{1}{\mathbf{Tr}}\left(\sigma^1\sigma^1\right)\cdot\underset{2}{\mathbf{Tr}}\left(\sigma^2\sigma^2\right)$$

Work out the trace brute force

$$\frac{1}{2}\underset{1}{\mathbf{Tr}}\left(\sigma^1\sigma^1\right)=\frac{1}{2}\underset{1}{\mathbf{Tr}}\left(\sigma_x^1\sigma_x^1\hat{\mathbf{x}}\hat{\mathbf{x}}+\sigma_y^1\sigma_y^1\hat{\mathbf{y}}\hat{\mathbf{y}}+\sigma_z^1\sigma_z^1\hat{\mathbf{z}}\hat{\mathbf{z}}\right)$$

$$\frac{1}{2}\underset{1}{\mathbf{Tr}}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\hat{\mathbf{x}}\hat{\mathbf{x}}+\begin{pmatrix}0&i\\-i&0\end{pmatrix}\begin{pmatrix}0&i\\-i&0\end{pmatrix}\hat{\mathbf{y}}\hat{\mathbf{y}}+\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}_z^1\hat{\mathbf{z}}\hat{\mathbf{z}}\right)$$

$$=\frac{1}{2}\underset{1}{\mathbf{Tr}}\begin{pmatrix}1&0\\0&1\end{pmatrix}\left(\hat{\mathbf{x}}\hat{\mathbf{x}}+\hat{\mathbf{y}}\hat{\mathbf{y}}+\hat{\mathbf{z}}\hat{\mathbf{z}}\right)=\left(\hat{\mathbf{x}}\hat{\mathbf{x}}+\hat{\mathbf{y}}\hat{\mathbf{y}}+\hat{\mathbf{z}}\hat{\mathbf{z}}\right)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\equiv\underline{\underline{\mathbf{U}}}$$

Note $\underline{\underline{\mathbf{U}}}\cdot\underline{\underline{\mathbf{U}}}=\left(\hat{\mathbf{x}}\hat{\mathbf{x}}+\hat{\mathbf{y}}\hat{\mathbf{y}}+\hat{\mathbf{z}}\hat{\mathbf{z}}\right)\cdot\left(\hat{\mathbf{x}}\hat{\mathbf{x}}+\hat{\mathbf{y}}\hat{\mathbf{y}}+\hat{\mathbf{z}}\hat{\mathbf{z}}\right)=\left(\hat{\mathbf{x}}\hat{\mathbf{x}}+\hat{\mathbf{y}}\hat{\mathbf{y}}+\hat{\mathbf{z}}\hat{\mathbf{z}}\right)=\underline{\underline{\mathbf{U}}}$

So we get my Eq.(7)

$$\left\langle\sigma^1\sigma^2\right\rangle=-\frac{1}{4}\underset{1}{\mathbf{Tr}}\left(\sigma^1\sigma^1\right)\cdot\underset{2}{\mathbf{Tr}}\left(\sigma^2\sigma^2\right)=-\underline{\underline{\mathbf{U}}}\cdot\underline{\underline{\mathbf{U}}}=\underline{\underline{\mathbf{U}}}$$

Finally

$$E(a,b)=-\mathbf{a}\cdot\left\langle\sigma^1\sigma^2\right\rangle\cdot\mathbf{b}=-\mathbf{a}\cdot\underline{\underline{\mathbf{U}}}\cdot\mathbf{b}$$

$$\mathbf{a}\cdot\left(\hat{\mathbf{x}}\hat{\mathbf{x}}+\hat{\mathbf{y}}\hat{\mathbf{y}}+\hat{\mathbf{z}}\hat{\mathbf{z}}\right)\cdot\mathbf{b}=a_xb_x+a_yb_y+a_zb_z=\mathbf{a}\cdot\mathbf{b}$$

The useful expression is $\frac{1}{2}\underset{1}{\mathbf{Tr}}\left(\sigma^1\sigma^1\right)=\underline{\underline{\mathbf{U}}}$

The correlation calculated is the same as GHSZ Eq.(6).

$$E(a,b)=\left\langle \sigma_a^1\sigma_b^2\rho_{12}\right\rangle=\underset{12}{\mathrm{Tr}}\left(\sigma_a^1\sigma_b^2\rho_{12}\right)=\underset{12}{\mathrm{Tr}}\left(\sigma_a^1\sigma_b^2\left|\Psi_{12}\right\rangle\left\langle\Psi_{12}\right|\right)$$
$$=\left\langle\Psi_{12}\right|\sigma_a^1\sigma_b^2\left|\Psi_{12}\right\rangle=\left\langle\Psi_{12}\right|\sigma^1\cdot\mathbf{a}\sigma^2\cdot\mathbf{b}\left|\Psi_{12}\right\rangle$$

Irr. Reps of the rotation group, SO(3). It turns out that any n-tuple of vectors $(r)^n$ where *r* is a vector is 3D Cartesian space, has irr reps that can be expressed in terms of many $\underline{\underline{\mathbf{U}}}$'s and at most one $\underline{\underline{\underline{\varepsilon}}}$ for example the dyadic, my Eq.(11) is

$$\sigma\sigma=\underline{\underline{\mathbf{U}}}+i\underline{\underline{\underline{\varepsilon}}}\cdot\sigma$$

And in component form, has the usual well know form, see any ref on Pauli spin properties.

$$\sigma_j\sigma_k=\left(\underline{\underline{\mathbf{U}}}\right)_{jk}+i\left(\underline{\underline{\underline{\varepsilon}}}\cdot\sigma\right)_{jk}=\delta_{jk}+i\varepsilon_{jkl}\sigma_l$$

The first term give polarizations and the second coherence.

I hope this helps. I will work out anything else people want for any equation in my papers.